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COMMUTATIVITY PRESERVING MAPS REVISITED

BY

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Dedicated to the memory of Kostia Beidar

ABSTRACT

K. I. Beidar and Y.-F. Lin have recently showed that under appropriate conditions a commutativity preserving map between (Jordan) algebras $\mathcal{A}$ and $\mathcal{Q}$ is of a standard form, unless it sends a certain subset of $\mathcal{A}$, which one could describe (unless $\mathcal{A}$ is very special) as a "large" one, into the center of $\mathcal{Q}$. We give a supplement to this statement by showing that this set often contains a nonzero ideal. In particular this makes it possible for us to give the definitive description of commutativity preservers in simple rings, as well as in prime rings provided that the map in question preserves commutativity in both directions.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be rings. We say that a map $\alpha\colon \mathcal{A} \to \mathcal{B}$ preserves commutativity if x^α and y^α commute whenever x and y commute. The problem of describing the form of such a map has been studied by a number of authors over the last thirty years. Starting with the paper by Watkins [24] it has been first treated in a series of linear algebraic papers in the context of algebras of matrices over fields. In the 80's these results have been extended to various operator algebras, and finally in the 90's the treatment has moved to ring theory. We refer the reader to two recent papers [8, 20] for more references and historic details.

The first ring-theoretic result was obtained by the present author [10, Theorem 2]. This result shows that the only bijective commutativity preserving linear

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maps between centrally closed prime algebras are, under certain mild technical assumptions, only the obvious ones, namely those that can be expressed as sums of scalar multiplies of (anti)isomorphisms and maps with the range in the center. In fact, instead of assuming that α preserves commutativity, only the condition that

$$[(x^2)^\alpha, x^\alpha] = 0 \tag{1}$$

holds for every $x \in \mathcal{A}$ was assumed (here, $[u,v]$ denotes $uv - vu$). This condition is of course weaker since x and x^2 certainly commute. From then on (1) has been studied in several ring-theoretic papers, and usually one refers (slightly inaccurately) to maps satisfying (1) as to commutativity preserving ones (incidentally, such maps actually do not always preserve commutativity [13, Example 2.1], but as many results show, quite often they do). Let us point out that (1) deals only with squares of elements in $\mathcal{A}$, and so this condition makes sense if $\mathcal{A}$ is a Jordan ring. On the other hand, the concept of a commutativity preserving map is essentially a Lie ring-theoretic one (in particular it generalizes the concept of a Lie homomorphism). Therefore, the interplay between the Jordan, Lie, and associative structures naturally appears in the study of these maps.

One can view (1) as a special example of a functional identity, and the result in [10] was obtained as an application of the theorem treating a more general functional identity. In fact, [10] was the first paper in which applications of the theory of functional identities were established. Later this theory has been extensively developed, and culminated in the works by Beidar and Chebotar on d-free sets [6, 7]. We will also use results on functional identities and d-free sets. For an introductory account on this topic we refer to [11]. On the other hand, a consensed survey, more than sufficient for our purposes, is contained in Section 2 of the recent article [8]. This work by Beidar and Lin has been the main inspiration for the present paper.

The starting-point of [8] is an ingenious example showing that [10, Theorem 2] does not necessarily hold for prime algebras that are not centrally closed [8, Example 1.2]. Then the authors considered maps α: $\mathcal{A} \to \mathcal{Q}$ satisfying (1) in a very general setting where $\mathcal{A}$ is any algebra (or a Jordan subalgebra of an algebra) and $\mathrm{Im}(\alpha)$ is a 5-free subset of a unital algebra $\mathcal{Q}$. Finally, they applied the obtained results to the prime algebra context. They showed that if α maps a prime algebra $\mathcal{A}$ onto a prime algebra $\mathcal{B}$, then, under certain technical assumptions, it must be of a standard form, unless α maps $\mathcal{L} = [[\mathcal{A},\mathcal{A}],\mathcal{A}] \circ [[\mathcal{A},\mathcal{A}],\mathcal{A}]$ into the center of $\mathcal{B}$ [8, Theorem 1.3] (here, $u \circ v$ denotes $uv + vu$).

A similar result was obtained for maps between symmetric elements of prime algebras with involution [8, Theorem 1.4].

The work on the present paper begun by observing that the map from the example by Beidar and Lin sends into the center a nonzero ideal, while the set $\mathcal{L}$ from the theorem is "only" a Lie ideal. So, can the result by Beidar and Lin be extended in such a way that the role of $\mathcal{L}$ would be replaced by an ideal? We shall see that this is indeed true. First we will, in Section 2, consider the general situation, and establish the key result, Theorem 2.4. Using this theorem we will, in Section 3, extend the results by Beidar and Lin on prime algebras [8, Theorems 1.3 and 1.4] by finding, under appropriate assumptions, nonzero ideals that α sends into the center (Corollaries 3.4 and 3.9). Corollaries 3.6 and 3.10 illustrate the significance of our approach. They give the definitive conclusion on α in two cases: in case of simple rings, and in case α preserves commutativity in both directions (i.e., $[x, y] = 0$ if and only if $[x^\alpha, y^\alpha] = 0$). We remark that the latter condition has also been studied extensively by a number of authors, but only in some rather concrete algebras appearing in linear algebra or operator theory. Therefore their characterization in general prime rings should be of some interest.

Although our main intention is to give a supplement to [8], from the technical point of view the present paper is almost independent of the paper by Beidar and Lin. Namely, the general context in which we shall work in the next section is slightly different from the one in [8]. On the one hand, this makes it possible for us to approach the problem more directly, and on the other hand, to analyze the prime ring case more carefully which yields somewhat better results with respect to PI rings of low degrees.

2. The general case

The concept of a d-free subset of a ring will play a crucial role in the present paper. To recall the definition, we have to introduce some notation. Let $\mathcal{Q}$ be a unital (associative) ring with center $\mathcal{C}$, and let $\mathcal{R}$ be a nonempty subset of $\mathcal{Q}$. By $\mathcal{R}^m$, where m is a positive integer, we denote the Cartesian product of m copies of $\mathcal{R}$; for convenience we also define $\mathcal{R}^0 = \{0\}$. For elements $x_1, x_2, \ldots, x_m \in \mathcal{R}$ we shall write

$$\overline{x}_m = (x_1, \ldots, x_m) \in \mathcal{R}^m.$$

Further, for any $1 \leq i \leq m$ we set

$$\overline{x}_m^i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) \in \mathcal{R}^{m-1},$$

and for $1 \le i < j \le m$ we set

$$\overline{x}_m^{ij} = \overline{x}_m^{ji} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1} \ldots, x_m) \in \mathcal{R}^{m-2}.$$

Let $\mathcal{I}$ and $\mathcal{J}$ be finite subsets of the set of all positive integers, and let m be such that $\mathcal{I}, \mathcal{J} \subseteq \{1, \ldots, m\}$. Further, let $E_i, F_j \colon \mathcal{R}^{m-1} \to \mathcal{Q}$, $i \in \mathcal{I}$, $j \in \mathcal{J}$, be arbitrary functions. The basic functional identities, upon which the theory of d-free sets is based, are:

$$\sum_{i\in\mathcal{I}} E_i(\overline{x}_m^i)x_i + \sum_{j\in\mathcal{J}} x_j F_j(\overline{x}_m^j) = 0 \quad \text{for all } \overline{x}_m \in \mathcal{R}^m, \tag{2}$$

$$\sum_{i\in\mathcal{I}} E_i(\overline{x}_m^i)x_i + \sum_{j\in\mathcal{J}} x_j F_j(\overline{x}_m^j) \in \mathcal{C} \quad \text{for all } \overline{x}_m \in \mathcal{R}^m. \tag{3}$$

The case when $\mathcal{I}$ or $\mathcal{J}$ is $\emptyset$, is not excluded. In such case it should be understood that a sum over $\emptyset$ is 0.

A natural possibility when (2) (and hence also (3)) holds is when the E_i's and F_j's are of the following form:

$$\begin{aligned} E_i(\overline{x}_m^i) &= \sum_{\substack{j\in\mathcal{J},\\ j\ne i}} x_j p_{ij}(\overline{x}_m^{ij}) + \lambda_i(\overline{x}_m^i), \quad i \in \mathcal{I}, \\ F_j(\overline{x}_m^j) &= -\sum_{\substack{i\in\mathcal{I},\\ i\ne j}} p_{ij}(\overline{x}_m^{ij})x_i - \lambda_j(\overline{x}_m^j), \quad j \in \mathcal{J}, \\ \lambda_k &= 0 \quad \text{if} \quad k \notin \mathcal{I} \cap \mathcal{J}, \end{aligned} \tag{4}$$

where $p_{ij} \colon \mathcal{R}^{m-2} \to \mathcal{Q}$, $i \in \mathcal{I}$, $j \in \mathcal{J}$, $i \ne j$, and $\lambda_k \colon \mathcal{R}^{m-1} \to \mathcal{C}$, $k \in \mathcal{I} \cup \mathcal{J}$ are arbitrary functions. Indeed, one can easily check that (4) is a solution of (2) (and hence also of (3)). We say that every solution of the form (4) is a standard solution of (2) (and of (3)). For example, the (only) standard solution of the functional identities $\sum_{i\in\mathcal{I}} E_i(\overline{x}_m^i) = 0$ and $\sum_{i\in\mathcal{I}} E_i(\overline{x}_m^i) \in \mathcal{C}$ is $E_i = 0$ for each $i \in \mathcal{I}$ (indeed, just consider the case when $\mathcal{J} = \emptyset$). The d-freeness is defined through the requirement that (2) and (3) have only standard solutions, provided that the number of variables in these functional identities is small enough. A precise definition reads as follows: We say that $\mathcal{R}$ is a d-**free** subset of $\mathcal{Q}$, where d is a positive integer, if for all finite subsets $\mathcal{I}, \mathcal{J}$ of the set of all positive integers, and for every m such that $\mathcal{I}, \mathcal{J} \subseteq \{1, 2, \ldots, m\}$, the following two conditions are satisfied:

(a) If $\max\{|\mathcal{I}|, |\mathcal{J}|\} \le d$, (2) implies (4).
(b) If $\max\{|\mathcal{I}|, |\mathcal{J}|\} \le d - 1$, (3) implies (4).

For more details concerning this concept we refer the reader to [6]. The definition of a d-free set is indeed a very technical one, but it has proved to be extremely useful. One of the main reasons is that many important subsets of various rings have turned out to be d-free. The basic result of this kind states that a prime ring $\mathcal{B}$ is a d-free subset of its maximal right (or left) ring of quotients $\mathcal{Q}$, unless $\mathcal{B}$ satisfies St_{2d-2}, the standard polynomial identity of degree $2d-2$. This is explicitly stated in [6], but essentially proved in Beidar's path-breaking paper [2].

We shall now fix the notation that will be used throughout this section.

By $\mathcal{J}$ we denote an arbitrary Jordan ring. The product in $\mathcal{J}$ will be denoted by $\circ$, and we shall write

$$[x, y, z] = (x \circ y) \circ z - x \circ (y \circ z)$$

for the associator of $x, y, z \in \mathcal{J}$. We shall use the same symbol, $\circ$, for denoting the Jordan product in an associative ring (i.e. $u \circ v = uv + vu$). This is of course a slight abuse of notation; on the other hand, if $\mathcal{J}$ is a special Jordan ring (and this is the case in which we are primarily interested), this notation is spotless. Let us also point that by x^2, where $x \in \mathcal{J}$, we mean $x \circ x$, so in case $\mathcal{J}$ is a special Jordan ring this does not coincide with the square of x with respect to the associative product.

By $\mathcal{U} \circ \mathcal{V}$, where $\mathcal{U}$ and $\mathcal{V}$ are subsets of $\mathcal{J}$, we denote the additive subgroup generated by all elements of the form $u \circ v$, $u \in \mathcal{U}$, $v \in \mathcal{V}$. Similar conventions apply to various other additive subgroups generated by certain sets.

Next, by $\mathcal{Q}$ we denote a unital (associative) ring such that its center $\mathcal{C}$ is a field; the latter is not a usual assumption in the theory of functional identities, but it is fulfilled in the case in which we are primarily interested (i.e., when $\mathcal{Q}$ is the maximal (right) ring of quotients of a prime ring and so $\mathcal{C}$ is the extended centroid), and it will enable us to simplify the arguments. Moreover, we shall assume that $\mathrm{char}(\mathcal{C})$, the characteristic of $\mathcal{C}$, is not 2. For $q, r \in \mathcal{Q}$ we shall write $q \equiv r$ if $q - r \in \mathcal{C}$.

Finally, α: $\mathcal{J} \to \mathcal{Q}$ will be an additive map satisfying (1) for every $x \in \mathcal{J}$, and we assume that $\mathrm{Im}(\alpha)$ is a 3-free subset of $\mathcal{Q}$.

The above setting is similar, but not the same to the one considered in [8]. One difference is that we added the assumption that $\mathcal{C}$ is a field (and so, unlike in [8], we are unable to cover the semiprime ring case). On the other hand, we require "only" the 3-freeness of $\mathrm{Im}(\alpha)$ while in [8] the 5-freeness was needed most of the time. Further, in [8] only the case when $\mathcal{J}$ is a special Jordan ring was treated, but the arguments seem to work for general Jordan rings. As a matter of fact, [8] deals with algebras over commutative rings, but we have decided to work in the context of rings.

Clearly (1) is fulfilled if α is a Jordan homomorphism, i.e., $(x \circ y)^\alpha = x^\alpha \circ y^\alpha$ holds for all $x, y \in \mathcal{J}$. More generally, (1) is fulfilled if α is of the form

$$x^\alpha = \gamma x^\beta + \xi(x),$$

where $\gamma \in \mathcal{C}$, $\beta\colon \mathcal{J} \to \mathcal{Q}$ is a Jordan homomorphism and $\xi\colon \mathcal{J} \to \mathcal{C}$ is an additive map. In such a case we shall say that α is of a **standard form**.

Our ultimate goal is to show that under appropriate assumptions α must necessarily be of a standard form. The first standard step in proving this is to show that there exist $\lambda \in \mathcal{C}$, an additive map $\mu\colon \mathcal{J} \to \mathcal{C}$ and a symmetric biadditive map $\tau\colon \mathcal{J} \times \mathcal{J} \to \mathcal{C}$ such that

$$(5) \qquad (x \circ y)^\alpha = \lambda x^\alpha \circ y^\alpha + \mu(x) y^\alpha + \mu(y) x^\alpha + \tau(x, y) \quad \text{for all } x, y \in \mathcal{J}.$$

Relying heavily on the assumption that $\mathrm{Im}(\alpha)$ is 3-free this follows easily from the general theory of functional identities, see [8, Lemma 3.1]. The next natural step is computing $((x^2 \circ y) \circ x)^\alpha = (x^2 \circ (y \circ x))^\alpha$ in two different ways (of course these two expressions coincide in view of the Jordan ring axiom). This was done already in [8, p. 1033]. For the sake of completeness, let us outline these calculations. Using (5) twice we get

$$\begin{aligned}((x^2 \circ y) \circ x)^\alpha \equiv{}& \lambda^2((x^2)^\alpha \circ y^\alpha) \circ x^\alpha + \lambda\mu(x)(x^2)^\alpha \circ y^\alpha + \lambda\mu(y)(x^2)^\alpha \circ x^\alpha \\ &+ \lambda\mu(x^2) x^\alpha \circ y^\alpha + \mu(x)\mu(y)(x^2)^\alpha \\ &+ \mu(x)\mu(x^2) y^\alpha \{2\lambda\tau(x^2, y) + \mu(x^2 \circ y)\} x^\alpha,\end{aligned}$$

and in a similar fashion we derive

$$\begin{aligned}(x^2 \circ (y \circ x))^\alpha \equiv{}& \lambda^2 (x^2)^\alpha \circ (y^\alpha \circ x^\alpha) + \lambda\mu(x)(x^2)^\alpha \circ y^\alpha + \lambda\mu(y)(x^2)^\alpha \circ x^\alpha \\ &+ \lambda\mu(x^2) x^\alpha \circ y^\alpha + \{\mu(x \circ y) + 2\lambda\tau(x, y)\}(x^2)^\alpha \\ &+ \mu(x)\mu(x^2) y^\alpha + \mu(x^2)\mu(y) x^\alpha.\end{aligned}$$

Comparing these two expressions and using (1) we obtain

$$\{\mu(x)\mu(y) - 2\lambda\tau(x, y) - \mu(x \circ y)\}(x^2)^\alpha \equiv \{\mu(x^2)\mu(y) - \mu(x^2 \circ y) - 2\lambda\tau(x^2, y)\} x^\alpha.$$

By (5) we have $(x^2)^\alpha = 2\lambda(x^\alpha)^2 + 2\mu(x)x^\alpha + \tau(x, x)$ and so the last relation can be rewritten as

$$(6) \qquad \varepsilon(x, y)(x^\alpha)^2 + \nu(x, x, y) x^\alpha \equiv 0 \quad \text{for all } x, y \in \mathcal{C},$$

where $\varepsilon\colon \mathcal{J} \times \mathcal{J} \to \mathcal{C}$ is a biadditive map given by

$$\varepsilon(x, y) = 2\lambda\{\mu(x)\mu(y) - 2\lambda\tau(x, y) - \mu(x \circ y)\},$$

and $\nu\colon \mathcal{J} \times \mathcal{J} \times \mathcal{J} \to \mathcal{C}$ is a triadditive map given by

$$\begin{aligned}\nu(x, x', y) = 2\lambda\{\tau(x \circ x', y) - 2\mu(x)\tau(x', y)\}\\ +\mu((x \circ x') \circ y) + 2\mu(x)\mu(x')\mu(y) - 2\mu(x \circ y)\mu(x') - \mu(x \circ x')\mu(y)\end{aligned}$$

A version of (6) was obtained also in [8]. From now on we proceed in a somewhat different way then in [8]. First we record a simple general observation on d-free sets.

LEMMA 2.1: *If $\mathcal{P}$ is an additive subgroup of $\mathcal{Q}$ such that every element in $\mathcal{P}$ is quadratic over $\mathcal{C}$ (i.e., algebraic of degree ≤ 2), then $\mathcal{P}$ is not a 3-free subset of $\mathcal{Q}$.*

Proof: If $x \in \mathcal{P} \setminus \mathcal{C}$ then there exists a unique $\tau(x) \in \mathcal{C}$ such that $x^2 \equiv \tau(x)x$. Set $\tau(x) = 2x$ for $x \in \mathcal{P} \cap \mathcal{C}$. We claim that $x \mapsto \tau(x)$ is an additive map from $\mathcal{P}$ into $\mathcal{C}$. One could establish this by using [12, Theorem 2.1]. Let us instead sketch a simple direct proof, using similar arguments as in the proof of [25, Theorem 3, p. 37]. We pick $u, v \in \mathcal{P}$ and substitute $u + v$, $u - v$, u, and v, respectively, for x in $x^2 \equiv \tau(x)x$; since $\mathrm{char}(\mathcal{C}) \neq 2$ it follows easily that $\tau(u+v) = \tau(u)+\tau(v)$ whenever $u, v, 1$ are $\mathcal{C}$-independent. So assume that $v = \alpha u+\beta$ for some $\alpha, \beta \in \mathcal{C}$. Using $v^2 \equiv \tau(v)v$ we infer that $\tau(v) = \alpha\tau(u)+2\beta$. Consequently, $\tau(u + v) = \tau((1 + \alpha)u + \beta) = (1 + \alpha)\tau(u) + 2\beta = \tau(u) + \tau(v)$, proving our claim.

Accordingly, $E\colon x \mapsto x - \tau(x)$ is an additive map from $\mathcal{P}$ into $\mathcal{Q}$ satisfying $E(x)x \in \mathcal{C}$ for all $x \in \mathcal{P}$, and therefore $E(x)y+E(y)x \in \mathcal{C}$ for all $x, y \in \mathcal{P}$. Now, if $\mathcal{P}$ was a 3-free subset of $\mathcal{Q}$, then by the very definition it would follow that $E = 0$, that is to say, $\mathcal{P} \subseteq \mathcal{C}$. But a subset of $\mathcal{C}$ of course cannot be 3-free. ∎

The argument in the next lemma is sligthly different from those used at similar places in [10] and [8]. This argument, which was suggested to us by Maja Fošner, in particular makes it possible for us to get rid of the assumption that $\mathcal{C} \neq GF(3)$ in [10, Theorem 2] (see Corollary 3.7 below).

LEMMA 2.2: *If $\lambda \neq 0$ then α is of a standard form.*

Proof: We first note that the complete linearization of (6) gives

$$\begin{aligned}&\varepsilon(x,y)u^\alpha\circ v^\alpha+\varepsilon(u,y)x^\alpha\circ v^\alpha+\varepsilon(v,y)x^\alpha\circ u^\alpha+(\nu(x,u,y)+\nu(u,x,y))v^\alpha\\ &\quad+(\nu(x,v,y)+\nu(v,x,y))u^\alpha+(\nu(v,u,y)+\nu(u,v,y))x^\alpha\equiv 0\\ &\qquad\text{for all } x,y,u,v\in\mathcal{J}.\end{aligned}\tag{7}$$

By Lemma 2.1 there exists $x\in\mathcal{J}$ such that x^α is not quadratic over $\mathcal{C}$. Therefore $\varepsilon(x,y)=\nu(x,x,y)=0$ for all $y\in\mathcal{J}$ by (6). Setting $x=u$ and $y=v$ in (7) we thus get $\varepsilon(y,y)(x^\alpha)^2\in\mathcal{C}x^\alpha+\mathcal{C}$, which in turn implies $\varepsilon(y,y)=0$ for all $y\in\mathcal{J}$. Since $\lambda\neq 0$ this means that $\mu(y)^2=2\lambda\tau(y,y)+\mu(y^2)$. Linearizing we get $\mu(x)\mu(y)=2\lambda\tau(x,y)+\mu(x\circ y)$. Accordingly, one can directly check, by using (5), that the map β: $\mathcal{J}\to\mathcal{Q}$ given by

$$x^\beta=\lambda x^\alpha+\mu(x)/2$$

is a Jordan homomorphism. One completes the proof by defining $\gamma=\lambda^{-1}$ and $\xi(x)=-\frac{1}{2}\lambda^{-1}\mu(x)$. ∎

Let $\mathcal{I}$ be the ideal of $\mathcal{J}$ generated by $[\mathcal{J},[\mathcal{J},\mathcal{J},\mathcal{J}]\circ[\mathcal{J},\mathcal{J},\mathcal{J}],\mathcal{J}]$, and let $\mathcal{L}$ be the ideal of $\mathcal{J}$ generated by $\mathcal{I}\circ\mathcal{I}$. The next lemma is of crucial importance; the novelties that this paper brings depend heavily upon it.

Lemma 2.3: *If $\lambda=0$ then $\mathcal{L}^\alpha\equiv 0$.*

Proof: Since $\lambda=0$, (6) reduces to $\nu(x,x,y)x^\alpha\equiv 0$. Therefore, for each $x\in\mathcal{J}$ we have either $x^\alpha\equiv 0$ or $\nu(x,x,y)=0$ for all $y\in\mathcal{J}$. We claim that the latter condition actually holds for every $x\in\mathcal{J}$. If this was not true, then there would be $x_0,y_0\in\mathcal{J}$ such that $\nu(x_0,x_0,y_0)\neq 0$ (and so $x_0^\alpha\equiv 0$). On the other hand, since $\mathrm{Im}(\alpha)$ is 3-free there exists $x_1\in\mathcal{J}$ such that $x_1^\alpha\not\equiv 0$, and so also $(x_0+x_1)^\alpha\not\equiv 0$ and $(x_0-x_1)^\alpha\not\equiv 0$. Consequently, $\nu(x_1,x_1,y_0)=0$, $\nu(x_0+x_1,x_0+x_1,y_0)=0$ and $\nu(x_0-x_1,x_0-x_1,y_0)=0$. But these relations contradict $2\nu(x_0,x_0,y_0)\neq 0$.

Thus $\nu(x,x,y)=0$ for all $x,y\in\mathcal{J}$ which can be, since $\lambda=0$, rewritten as

$$\mu(x^2\circ y)=2\mu(x\circ y)\mu(x)+\mu(x^2)\mu(y)-2\mu(x)^2\mu(y).$$

Linearizing we get

$$\begin{aligned}&\mu((x\circ z)\circ y)=\\ &\qquad\mu(x\circ y)\mu(z)+\mu(z\circ y)\mu(x)+\mu(x\circ z)\mu(y)-2\mu(x)\mu(z)\mu(y).\end{aligned}\tag{8}$$

Note that y and z appear symmetrically on the right hand side of (8). Therefore, the left hand side remains the same after changing the roles of y and z, that is to say, $\mu((x \circ z) \circ y) = \mu((x \circ y) \circ z)$, or equivalently $\mu([z, x, y]) = 0$. Setting $\mathcal{K} = \mathrm{Ker}(\mu)$ we thus have

$$[\mathcal{J}, \mathcal{J}, \mathcal{J}] \subseteq \mathcal{K}. \tag{9}$$

We claim that the ideal of $\mathcal{J}$ generated by $\mathcal{K} \cap (\mathcal{K} \circ \mathcal{K})$ is contained in $\mathcal{K}$. Pick $u \in \mathcal{K} \cap (\mathcal{K} \circ \mathcal{K})$. Writing $u = \sum_i x_i \circ z_i$ with $x_i, z_i \in \mathcal{K}$ and using (8) we see that $u \circ y \in \mathcal{K}$ for all $y \in \mathcal{J}$. Consequently, applying (9) it follows by induction on n that

$$\begin{aligned}
&(((\ldots((u \circ y_1) \circ y_2)\ldots) \circ y_{n-2}) \circ y_{n-1}) \circ y_n \\
&\qquad = [(\ldots((u \circ y_1) \circ y_2)\ldots) \circ y_{n-2}, y_{n-1}, y_n] \\
&\qquad\quad + ((\ldots((u \circ y_1) \circ y_2)\ldots) \circ y_{n-2}) \circ (y_{n-1} \circ y_n) \in \mathcal{K}
\end{aligned}$$

for all $u \in \mathcal{K} \cap (\mathcal{K} \circ \mathcal{K})$, $y_i \in \mathcal{J}$, $n \geq 2$, proving our claim.

Next, given $x, y \in \mathcal{J}$ and $u_1, u_2 \in \mathcal{K}$ we have

$$[x, u_1 \circ u_2, z] = [x, u_1, z] \circ u_2 + u_1 \circ [x, u_2, z] \in \mathcal{K} \cap (\mathcal{K} \circ \mathcal{K})$$

(here we used the well-known fact that the map $u \mapsto [x, u, z]$ is a derivation). That is, $[\mathcal{J}, \mathcal{K} \circ \mathcal{K}, \mathcal{J}] \subseteq \mathcal{K} \cap (\mathcal{K} \circ \mathcal{K})$, and so, in particular,

$$[\mathcal{J}, [\mathcal{J}, \mathcal{J}, \mathcal{J}] \circ [\mathcal{J}, \mathcal{J}, \mathcal{J}], \mathcal{J}] \subseteq \mathcal{K} \cap (\mathcal{K} \circ \mathcal{K}).$$

Therefore, by the preceding paragraph we have

$$\mathcal{I} \subseteq \mathcal{K}. \tag{10}$$

Since (5) shows that

$$(x \circ y)^\alpha \equiv \mu(x) y^\alpha + \mu(y) x^\alpha \quad \text{for all } x, y \in \mathcal{J}, \tag{11}$$

it follows that $(\mathcal{I} \circ \mathcal{I})^\alpha \equiv 0$. Furthermore, given $u, v \in \mathcal{I}$ we have $\mu(u \circ v) = 0$ by (10), then (11) yields that for every $y \in \mathcal{J}$ we have

$$((u \circ v) \circ y)^\alpha \equiv \mu(y)(u \circ v)^\alpha \equiv 0.$$

Thus $((\mathcal{I} \circ \mathcal{I}) \circ \mathcal{J})^\alpha = 0$. Using the linearized form of the Jordan ring axiom $(x^2 \circ y) \circ x = x^2 \circ (y \circ x)$ one easily infers that $\mathcal{L} = \mathcal{I} \circ \mathcal{I} + (\mathcal{I} \circ \mathcal{I}) \circ \mathcal{J}$. Consequently, $\mathcal{L}^\alpha \equiv 0$. ∎

The following theorem summarizes what has been established above.

THEOREM 2.4: *Let $\mathcal{J}$ be a Jordan ring, let $\mathcal{I}$ be the ideal of $\mathcal{J}$ generated by $\big[\mathcal{J}, [\mathcal{J},\mathcal{J},\mathcal{J}] \circ [\mathcal{J},\mathcal{J},\mathcal{J}], \mathcal{J}\big]$, and let $\mathcal{L}$ be the ideal of $\mathcal{J}$ generated by $\mathcal{I} \circ \mathcal{I}$. Further, let $\mathcal{Q}$ be a unital (associative) ring such that its center $\mathcal{C}$ is a field with $\mathrm{char}(\mathcal{C}) \neq 2$, and let $\alpha\colon \mathcal{J} \to \mathcal{Q}$ be an additive map satisfying $[(x^2)^\alpha, x^\alpha] = 0$ for all $x \in \mathcal{J}$. If $\mathrm{Im}(\alpha)$ is a 3-free subset of $\mathcal{Q}$, then either α is of a standard form or $\mathcal{L}^\alpha \subseteq \mathcal{C}$.*

As the referee pointed out, the class of maps satisfying the condition that we consider, i.e. $[(x^2)^\alpha, x^\alpha] = 0$ for all $x \in \mathcal{J}$, includes also, by the very definition, Jordan representations. Moreover, Theorem 2.4 seems to be new even for them. Incidentally, (8) shows that the (central) map μ is a Jordan representation.

3. Some special Jordan rings

In this section we shall examine the ideal $\mathcal{L}$ in some particular Jordan rings $\mathcal{J}$ (in fact, in special ones, i.e.,those that are Jordan subrings of associative rings), and thereby obtain various corollaries to Theorem 2.4.

We shall assume throughout the section that all rings we are dealing with have characteristic different from 2; we shall use this without further mention, except in formulations of the main results.

Let us first point out that if $\mathcal{L} = \mathcal{J}$, then $\mathcal{L}^\alpha \subseteq \mathcal{C}$ cannot occur (since $\mathrm{Im}(\alpha)$ is a 3-free set), and so α is necessarily of a standard form. As another extreme, if $\mathcal{L} = 0$ then Theorem 2.4 does not give any useful information. This can indeed occur but only in some very special cases. Let us first give a few comments about this.

3.1. QUADRATIC JORDAN ALGEBRAS. Suppose that $\mathcal{J}$ is quadratic over a field $\mathcal{F}$, that is, $\mathcal{J}$ is a Jordan algebra over $\mathcal{F}$ and there exists a linear functional τ on $\mathcal{J}$ such that $x^2 - \tau(x)x \in \mathcal{F}\mathbf{1}$ for all $x \in \mathcal{J}$. If $\mathcal{Q}$ is any algebra over $\mathcal{F}$, then every $\mathcal{F}$-linear map $\alpha\colon \mathcal{J} \to \mathcal{Q}$ that sends $\mathbf{1}$ into the center of $\mathcal{Q}$ satisfies $[(x^2)^\alpha, x^\alpha] = 0$ for all $x \in \mathcal{J}$, and of course there is no reason why it should be either of a standard form or why it should send a nonzero ideal into the center of $\mathcal{Q}$. The next lemma therefore does not come as a surprise. The referee pointed out to us that this lemma is well-known and is contained for example in [16] and [23]; but since the proof is short and straightforward, we include it anyway.

LEMMA 3.1: *If $\mathcal{J}$ is a quadratic Jordan algebra, then $[\mathcal{J},\mathcal{J},\mathcal{J}]\circ[\mathcal{J},\mathcal{J},\mathcal{J}] \subseteq \mathcal{F}\mathbf{1}$ (and therefore the ideals $\mathcal{I}$ and $\mathcal{L}$ from Theorem 2.4 are both zero).*

Proof: Note that every $x \in \mathcal{J}$ can be written as $x = \alpha\mathbf{1} + x'$ where $\alpha \in \mathcal{F}$ and $\tau(x') = 0$. Hence we see that every associator $[x, y, z]$ can be represented as $[x', y', z']$ with $\tau(x') = \tau(y') = \tau(z') = 0$. From the linearized form of $x^2 - \tau(x)x \in \mathcal{F}\mathbf{1}$, that is

$$2x \circ y - \tau(x)y - \tau(y)x \in \mathcal{F}\mathbf{1}, \tag{12}$$

we see that $x' \circ y' \in \mathcal{F}\mathbf{1}$ and $y' \circ z' \in \mathcal{F}\mathbf{1}$. Thus, $[x, y, z] = \lambda x' + \mu z'$ with $\lambda, \mu \in \mathcal{F}$, which shows that $\tau([x, y, z]) = 0$. The desired conclusion therefore follows immediately from (12). ∎

Now assume that $\mathcal{J}$ is any Jordan algebra over $\mathcal{F}$ that has a quadratic Jordan algebra $\mathcal{J}_0$ as a homomorphic image. Further, if $\mathcal{Q}$ is any associative $\mathcal{F}$-algebra and $\alpha_0\colon \mathcal{J}_0 \to \mathcal{Q}$ is a linear map sending $\mathbf{1}$ into the center of $\mathcal{Q}$, then $\alpha = \alpha_0\pi$, where π is a homomorphism of $\mathcal{J}$ onto $\mathcal{J}_0$, satisfying $[(x^2)^\alpha, x^\alpha] = 0$. This gives us a variety of examples of maps that are not of standard forms, but their kernels contain $\mathcal{L}$ (and $\mathcal{I}$).

The appeareance of the ideals $\mathcal{I}$ and $\mathcal{L}$ in Theorem 2.4 may seem a bit artificial at first glance, but now we have seen that there are good reasons for this.

3.2. Associative rings. In this subsection we will consider the case where $\mathcal{A}$ is an associative ring and $\mathcal{J} = \mathcal{A}^+$, i.e., $\mathcal{J}$ coincides with $\mathcal{A}$ as an additive group, and is equipped with the Jordan product $x \circ y = xy + yx$. Note that in this case we have

$$[x, y, z] = [[z, x], y].$$

Therefore, $\mathcal{I}$ is the Jordan ideal of $\mathcal{A}$ generated by

$$\big[[\mathcal{A}, \mathcal{A}], [[\mathcal{A}, \mathcal{A}], \mathcal{A}] \circ [[\mathcal{A}, \mathcal{A}], \mathcal{A}]\big],$$

and $\mathcal{L}$ is the Jordan ideal of $\mathcal{A}$ generated by $\mathcal{I} \circ \mathcal{I}$.

We begin with an elementary observation concerning the ring $\mathcal{A} = M_n(\mathcal{R})$ of $n \times n$ matrices, $n \geq 3$, over an arbitrary unital ring $\mathcal{R}$. By e_{ij} we denote the matrix units in $\mathcal{A}$. Note that for any three distinct i, j, k we have

$$u = \big[[e_{ik} + e_{ki}, e_{ii}], [[e_{ij} + e_{ji}, e_{ii}], e_{ii}] \circ [[e_{ij} + e_{ji}, e_{ii}], e_{ii}]\big] = 2(e_{ik} + e_{ki}) \tag{13}$$

and hence

$$8e_{ii} = e_{ii} \circ ((e_{ik} + e_{ki}) \circ u). \tag{14}$$

We shall apply these identities repeatedly, first in the proof of the next result.

Corollary 3.2: *Let $\mathcal{R}$, $\mathcal{S}$ be unital rings and assume that the center of $\mathcal{S}$ is a field of characteristic different from 2. Let $n \geq 3$, $m \geq 3$, and let $\alpha\colon M_n(\mathcal{R}) \to M_m(\mathcal{S})$ be a surjective additive map such that $[(x^2)^\alpha, x^\alpha] = 0$ for all $x \in M_n(\mathcal{R})$. Then α is of a standard form.*

Proof: Since $m \geq 3$, by [3, Corollary 5.12] we know that $M_m(\mathcal{S})$ is 3-free (as a subset of itself). Moreover, its center is a field by assumption and so the conditions of Theorem 2.4 are fulfilled.

Suppose that α was not of a standard form. Then $\mathcal{L}^\alpha$ would be contained in the center of $M_m(\mathcal{S})$. However, (14) shows that $\mathcal{I}$ contains all matrices of the form $8e_{ii}$, hence it contains $8id$ where id is the identity matrix, which in turn implies that $\mathcal{L}$ contains $8id \circ 8id = 2^7 id$. Accordingly, $2^8 M_n(\mathcal{R}) \subseteq \mathcal{L}$, implying that $2^8 M_m(\mathcal{S})$ is contained in the center of $M_m(\mathcal{S})$ — a contradiction. ∎

As already mentioned, the study of commutativity preserving maps originated in linear algebra, and so Corollary 3.2 generalizes some of these results. On the other hand, it partially improves [3, Corollary 6.8].

The case when $n = 2$ is really exceptional. Namely, if $\mathcal{R} = \mathcal{F}$ is a field, then $M_2(\mathcal{F})$ is quadratic over $\mathcal{F}$. Therefore $M_2(\mathcal{F})$ (as well as of some of its subrings) must really be excluded in general results. Recall that a prime ring $\mathcal{A}$ satisfies St_4, the standard polynomial identity of degree 4, if and only if $\mathcal{A}$ can be embedded into $M_2(\mathcal{F})$ for some field $\mathcal{F}$. Of course, in these rings $\mathcal{I}$ and $\mathcal{L}$ are both 0. The next lemma shows that the converse is also true.

Lemma 3.3: *Let $\mathcal{A}$ be a prime ring. If $\big[[\mathcal{A},\mathcal{A}], [[\mathcal{A},\mathcal{A}],\mathcal{A}] \circ [[\mathcal{A},\mathcal{A}],\mathcal{A}]\big] = 0$, then $\mathcal{A}$ satisfies St_4.*

Proof: Our assumption can be read as that

$$[[x_1, x_2], [[x_3, x_4], x_5] \circ [[x_6, x_7], x_8]] \tag{15}$$

is a polynomial identity on $\mathcal{A}$. By Posner's theorem (see e.g. [15, Section 1.4] or [21, Section 1.7]) the central closure $\mathcal{AC}$ of $\mathcal{A}$ is a finite dimensional central simple algebra over $\mathcal{C}$, the field of fractions of the center of $\mathcal{A}$. Accordingly, if $\overline{\mathcal{C}}$ is the algebraic closure of $\mathcal{C}$, then the $\overline{\mathcal{C}}$-algebra $\overline{\mathcal{A}} = \mathcal{AC} \otimes_{\mathcal{C}} \overline{\mathcal{C}}$ is isomorphic to $M_n(\overline{\mathcal{C}})$ for some $n \geq 1$. Of course, $\overline{\mathcal{A}}$ also satisfies (15). But then (13) implies that $n \leq 2$. Consequently, $\mathcal{A}$ satisfies St_4. ∎

Corollary 3.4: *Let $\mathcal{A}$ and $\mathcal{B}$ be prime rings, both of them of characteristic not equal to 2 and not satisfying St_4. If $\alpha\colon \mathcal{A} \to \mathcal{B}$ is a surjective additive map* *such that* $[(x^2)^\alpha, x^\alpha] = 0$ *for all* $x \in \mathcal{A}$*, then either* α *is of a standard form or there exists a nonzero ideal* $\mathcal{U}$ *of* $\mathcal{A}$ *such that* α *maps* $\mathcal{U}$ *into the center of* $\mathcal{B}$*.*

Proof: First of all, since $\mathcal{B}$ does not satisfy St_4, it is a 3-free subset of its maximal right ring of quotients $\mathcal{Q}$ [6, Theorem 2.4]. Thus the conditions of Theorem 2.4 are fulfilled (in this context $\mathcal{C}$ is the extended centroid of $\mathcal{B}$). Therefore, α is of a standard form unless $\mathcal{L}^\alpha \subseteq \mathcal{C}$; since α maps into $\mathcal{B}$ this is the same as saying that $\mathcal{L}^\alpha$ is contained in the center of $\mathcal{B}$.

By Lemma 3.3, $\mathcal{I} \neq 0$. Pick a nonzero $a \in \mathcal{I}$. Then $axa \neq 0$ for some $x \in \mathcal{A}$. Since

$$2axa = a \circ (a \circ x) - a^2 \circ x \tag{16}$$

it clearly follows that $\mathcal{I} \circ \mathcal{I} \neq 0$. That is, $\mathcal{L} \neq 0$. Finally, by a well-known theorem of Herstein [14, Theorem 1.1], $\mathcal{L}$ contains a nonzero ideal $\mathcal{U}$ of $\mathcal{A}$, and the proof is complete. ∎

Remark 3.5: We claim that if α in Corollary 3.2 is of a standard form then the Jordan homomorphism β, given in $x^\alpha = \gamma x^\beta + \xi(x)$, is either a homomorphism or an antihomomorphism. On the one hand, this can be easily established by modifying slightly the proof of a well-known Herstein's theorem [14, Theorem 3.1], and on the other hand, the referee remarked that one can also use the results in [17] and [22] to show this.

Corollary 3.6: *Under the assumptions of Corollary 3.2,* α *must necessarily be of a standard form provided that either*

(a) $\mathcal{A}$ *is a simple ring, or*

(b) α *preserves commutativity in both directions.*

Proof: Since α is surjective and $\mathcal{B}$ is noncommutative, the case when (a) holds is obvious. If α preserves commutativity in both directions, then α cannot map a nonzero ideal of $\mathcal{A}$ into the center of $\mathcal{B}$; namely, as a noncommutative prime ring $\mathcal{A}$ cannot contain commutative nonzero ideals. ∎

It is clear that in Corollary 3.2 the assumption that $\mathcal{A}$ does not satisfy St_4 is really necessary (even when α preserves commutativity, cf. [20, Theorem 1.1]). There does not seem to be, however, any good reason why the same should be required for the ring $\mathcal{B}$. Let us conclude this subsection by showing that at least in the situation considered already in [10] this assumption is indeed redundant.

Now assume that $\mathcal{A}$ is a unital algebra over a field $\mathcal{C}$. We say that $\mathcal{A}$ is centrally closed over $\mathcal{C}$ if the extended centroid of $\mathcal{A}$ (and hence also the center of $\mathcal{A}$) is equal to $\mathcal{C}\mathbf{1}$.

COROLLARY 3.7: *Let $\mathcal{A}$ and $\mathcal{B}$ be centrally closed prime algebras over a field $\mathcal{C}$ with* $\mathrm{char}(\mathcal{C}) \neq 2$, *and suppose that $\mathcal{A}$ does not satisfy St_4. If $\alpha\colon \mathcal{A} \to \mathcal{B}$ is a bijective linear map such that $[(x^2)^\alpha, x^\alpha] = 0$ for all $x \in \mathcal{A}$, then α is of a standard form.*

Proof: The key observation is that α satisfies (5) even when $\mathcal{B}$ satisfies St_4, and moreover, in this case we may take $\lambda = 0$. This follows by rewriting $[(x^2)^\alpha, x^\alpha] = 0$ as $[q(y), y] = 0$ for all $y \in \mathcal{B}$, where $q(y) = ((y^{\alpha^{-1}})^2)^\alpha$, and then applying [12, Theorem 3.1].

So, in any case (5) holds. Just as in [10, p. 535] one shows that α maps $\mathbf{1}$ into the center of $\mathcal{B}$ — all one has to do is to linearize $[(x^2)^\alpha, x^\alpha] = 0$ and set $\mathbf{1}$ at appropriate places. Since $\mathcal{A}$ and $\mathcal{B}$ are centrally closed over $\mathcal{C}$ and α is linear this implies that $(\mathcal{C}\mathbf{1})^\alpha = \mathcal{C}\mathbf{1}$. Consequently, by taking $y = x$ in (5) we see that λ cannot be 0; namely, otherwise every element in $\mathcal{A}$ would be quadratic over $\mathcal{C}$ which contradicts our assumption that $\mathcal{A}$ does not satisfy St_4. Therefore, $\lambda \neq 0$ so, in particular, $\mathcal{B}$ cannot satisfy St_4. But then Lemma 2.2 can be applied. ∎

Corollary 3.7 removes two unnecessary technical conditions in [10, Theorem 2]: the one that $\mathcal{B}$ does not satisfy St_4, and the one that $\mathcal{C} \neq GF(3)$. We remark that one can easily show a bit more about β in this case, namely that it is either an algebra isomorphism or an algebra antiisomorphism from $\mathcal{A}$ onto $\mathcal{B}$ (see [10, pp. 537–538]).

3.3. SYMMETRIC ELEMENTS. Let $\mathcal{A}$ be a ring with involution $*$. By $\mathcal{S}(\mathcal{A})$ we denote the set of its symmetric elements, i.e. $\mathcal{S}(\mathcal{A}) = \{x \in \mathcal{A} | x^* = x\}$. Of course, $\mathcal{S}(\mathcal{A})$ is a Jordan subring of $\mathcal{A}$. In this subsection we shall consider the case where α maps from $\mathcal{S}(\mathcal{A})$ onto $\mathcal{S}(\mathcal{B})$ where $\mathcal{B}$ is another ring with involution. We shall follow a similar pattern as in the previous subsection.

Recall that the symplectic involution on the ring $M_{2m}(\mathcal{F}) \cong M_2(M_m(\mathcal{F}))$ is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} d^{tr} & -b^{tr} \\ -c^{tr} & a^{tr} \end{bmatrix}$$

where $a, b, c, d \in \mathcal{M}_m(\mathcal{F})$ and x^{tr} denotes the transpose of the matrix x. Thus, the set $\mathcal{S}$ of symmetric elements with respect to this involution consists of matrices of the form

$$\begin{bmatrix} a & k \\ l & a^{tr} \end{bmatrix} \tag{17}$$

where $a, k, l \in \mathcal{M}_m(\mathcal{F})$ with $k^{tr} = -k$, $l^{tr} = -l$. One can easily check that if $m = 2$ then $\mathcal{S}$ is quadratic over $\mathcal{F}$, and so in this case the conclusion of Lemma 3.1 holds. Therefore, the ring $M_4(\mathcal{F})$, and more generally, rings satisfying St_8, will play a special role in the sequel.

Lemma 3.8: *Let $\mathcal{A}$ be a prime ring with involution, and let $\mathcal{S} = \mathcal{S}(\mathcal{A})$. If $\big[[\mathcal{S},\mathcal{S}], [[\mathcal{S},\mathcal{S}],\mathcal{S}] \circ [[\mathcal{S},\mathcal{S}],\mathcal{S}]\big] = 0$, then $\mathcal{A}$ satisfies St_8.*

Proof: We are assuming that $\mathcal{S}$ satisfies the polynomial identity (15). It is well-known that the involution on $\mathcal{A}$ can be uniquely extended to the involution on the central closure $\mathcal{AC}$ of $\mathcal{A}$ (see e.g. [9, Proposition 2.5.4]), and of course $\mathcal{S}(\mathcal{AC})$ also satisfies (15). If $*$ is the involution of the second kind (i.e., $*$ is not the identity on $\mathcal{C}$), then $\mathcal{C}$ contains a nonzero element μ such that $\mu^* = -\mu$, hence every element in $\mathcal{AC}$ can be written as $s_1 + \mu s_2$ with $s_i \in \mathcal{S}(\mathcal{AC})$, it follows that $\mathcal{A}$ satisfies (15). Lemma 3.3 therefore tells us that $\mathcal{A}$ satisfies St_4, and the proof is complete in this case. So we may assume that $*$ is of the first kind (i.e., $\lambda^* = \lambda$ for all $\lambda \in \mathcal{C}$). Let $\overline{\mathcal{C}}$ be the algebraic closure of $\mathcal{C}$. We extend $*$ to the $\overline{\mathcal{C}}$-algebra $\overline{\mathcal{A}} = \mathcal{AC} \otimes_{\mathcal{C}} \overline{\mathcal{C}}$ by $(x \otimes \lambda)^* = x^* \otimes \lambda$. Then $\mathcal{S}(\overline{\mathcal{A}})$ still satisfies (15). Since $\mathcal{S}$ satisfies a polynomial identity, $\mathcal{A}$ is a PI ring (see [1] or [19]), and so as in the proof of Lemma 3.3 we may conclude that $\overline{\mathcal{A}} \cong M_n(\overline{\mathcal{C}})$ for some $n \geq 1$. If $*$ is the transpose involution on $\overline{\mathcal{A}}$ then we see from (13) that $n \leq 2$, so that $\mathcal{A}$ satisfies St_4 in this case. According to [9, Corollary 4.6.13] there is just one case that remains to be considered: $n = 2m$ and $*$ is the symplectic involution on $\overline{\mathcal{A}}$. Just by considering matrices of the form (17) with $k = l = 0$ we are then forced to conclude that $M_m(\overline{\mathcal{C}})$ satisfies (15), which by (13) implies that $m \leq 2$. Thus $\overline{\mathcal{A}} \cong M_2(\overline{\mathcal{C}})$ or $\overline{\mathcal{A}} \cong M_4(\overline{\mathcal{C}})$, and hence $\mathcal{A}$ satisfies St_8. ∎

Recall that an ideal $\mathcal{U}$ of a ring with involution is said to be a $*$-ideal if $\mathcal{U}^* = \mathcal{U}$.

Corollary 3.9: *Let $\mathcal{A}$ and $\mathcal{B}$ be prime rings with involution and of characteristic not 2. Suppose that $\mathcal{A}$ does not satisfy St_8, and that $\mathcal{B}$ does not satisfy St_{12}. If α: $\mathcal{S}(\mathcal{A}) \to \mathcal{S}(\mathcal{B})$ is a surjective additive map such that $[(x^2)^\alpha, x^\alpha] = 0$ for all $x \in \mathcal{S}(\mathcal{A})$, then either α is of a standard form or there exists a nonzero $*$-ideal $\mathcal{U}$ of $\mathcal{A}$ such that α maps $\mathcal{S}(\mathcal{U})$ into the center of $\mathcal{B}$.*

Proof: Since $\mathcal{B}$ does not satisfy St_{12}, $\mathcal{S}(\mathcal{B})$ is a 3-free subset of its maximal right ring of quotients $\mathcal{Q}$ (see [6, Lemma 2.2] and [5, Theorem 2.4]). Thus we are in a position to apply Theorem 2.4. Assuming that α is not of a standard form it follows that $\mathcal{L}^\alpha$ is contained in the center of $\mathcal{B}$. It remains to show that $\mathcal{L}$ contains $\mathcal{S}(\mathcal{U})$ for some nonzero $*$-ideal $\mathcal{U}$ of $\mathcal{A}$. In view of Herstein's result [15, Theorem 2.1.12] it suffices to show that $\mathcal{L} \neq 0$.

Lemma 3.8 tells us that $\mathcal{I} \neq 0$. Given a nonzero $a \in \mathcal{I}$, there exists $x \in \mathcal{S}$ such that $axa \neq 0$ [18, Lemma 3.1]. Therefore (16) shows that $\mathcal{I} \circ \mathcal{I} \neq 0$. That is, $\mathcal{L} \neq 0$. ∎

For the case when α is of a standard form, Beidar and Lin have proved more, namely, that the Jordan homomorphism β can be extended to a homomorphism on the subring generated by $\mathcal{S}(\mathcal{A})$ [8, Theorem 1.4]. We shall not consider this matter here.

Corollary 3.10: *Under the assumptions of Corollary 3.9, α must necessarily be of a standard form provided that either*

(a) *$\mathcal{A}$ is a simple ring, or*

(b) *α preserves commutativity in both directions.*

Proof: Since $\mathcal{B}$ does not satisfy St_{12}, $\mathcal{S}(\mathcal{B})$ is not contained in the center of $\mathcal{B}$. One can easily check this directly (actually, $\mathcal{S}(\mathcal{B})$ can be contained in the center only when $\mathcal{B}$ satisfies St_4); on the other hand, this also follows from Lemma 3.8. Thus, α must be of a standard form if (a) holds. One handles (b) in a similar fashion; at the end one has to apply the fact that a nonzero ideal of a prime ring is again a prime ring which satisfies the same standard polynomial identities as the ring itself. ∎

Possibly the assumption in Corollary 3.9 that $\mathcal{B}$ should not satisfy St_{12} is redundant. However, we do not have tools avaliable to establish (at least) an analogue of Corollary 3.7. Let us just remark that our method shows that the assumption in [4, Theorem 3.1] that the characteristic should be different from 3 can be removed.

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